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S^1-actions on highly connected manifolds

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Abstract

We use the theory of elliptic genera to exhibit new obstructions to smooth non-trivial S^1-actions on highly connected manifolds.

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1. Introduction

Let M be a smooth closed connected *Spin*-manifold. A celebrated theorem of Atiyah and Hirzebruch [1] asserts that the index of the Dirac operator on M, the $\hat{A}$-genus, vanishes if M admits a smooth non-trivial S^1-action. Here smoothness of the action is necessary since there exist examples of topological S^1-actions on *Spin*-manifolds with non-zero $\hat{A}$-genus (cf. [4, p. 352], and [5]).

In this note we use the elliptic genus to show that additional obstructions exist if one restricts to highly connected manifolds. We recall that the elliptic genus is a bordism invariant which assigns to M a modular function $\Phi(M)$ for $\Gamma_0(2)$. In one of the cusps of $\Gamma_0(2)$ this modular function expands as a series of twisted signatures which, as explained by Witten [13], describes the "signature" of the free loop space $\mathcal{L}M$ of M localized at the constant loops. In a different cusp $\Phi(M)$ expands as a series of indices of twisted Dirac operators

$$\Phi_0(M) = q^{-\frac{\dim M}{8}} \cdot (\hat{A}(M) - \hat{A}(M, TM) \cdot q + \hat{A}(M, \Lambda^2 TM + TM) \cdot q^2 + \ldots).$$

Here $\hat{A}(M, E)$ denotes the index of the Dirac operator twisted with $E \otimes \mathbb{C}$.

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The main feature of the elliptic genus is its rigidity under S^1-actions conjectured by Witten and proved by Taubes and by Bott and Taubes (see [3] and references therein). In [9] Hirzebruch and Slodowy used the rigidity theorem to show that the coefficients of $\Phi_0(M)$ define obstructions to actions by involutions with large fixed point codimension.

In this note we show that their approach can also be used to define obstructions to the existence of S^1-actions on highly connected manifolds.

Theorem 1.1. *Let M be a k-connected manifold, $k \geq 4r > 0$. If M admits a smooth non-trivial S^1-action then the first $(r+1)$ coefficients of $\Phi_0(M)$ vanish.*

The note is structured in the following way. In the next section we review relevant properties of the elliptic genus. In Section 3 we prove a slightly more general version of Theorem 1.1. In the final section we show by example that Theorem 1.1 is independent of the vanishing theorem for the Witten genus [12,6].

2. Elliptic genera

In this section we review relevant properties of the elliptic genus (for more information see [11,8]). The elliptic genus of M is a modular function $\Phi(M)$ of weight 0 with $\mathbb{Z}_2$-character for

$$\Gamma_0(2) := \left\{ A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 2 \right\}.$$

In one of the cusps (the signature cusp) $\Phi(M)$ expands as a series of twisted signatures

$$\begin{aligned} \operatorname{sign}(q, \mathcal{L}M) &:= \operatorname{sign}\left(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM\right) \\ &= \operatorname{sign}(M) + 2 \cdot \operatorname{sign}(M, TM) \cdot q + \ldots = \sum_{n \geq 0} \operatorname{sign}(M, E_n) q^n \in \mathbb{Z}[\![q]\!]. \end{aligned}$$

Here $\Lambda_t := \sum_i \Lambda^i \cdot t^i$ (resp. $S_t := \sum_i S^i \cdot t^i$) denotes the exterior (resp. symmetric) power operation and $\operatorname{sign}(M, E)$ denotes the index of the signature operator twisted with the complexification $E \otimes \mathbb{C}$. Note that each E_n is a virtual complex vector bundle associated to TM.

Now assume S^1 acts smoothly on M. Then the twisted signatures occurring in $\operatorname{sign}(q, \mathcal{L}M)$ refine to virtual S^1-representations and the expansion refines to a series $\operatorname{sign}_{S^1}(q, \mathcal{L}M) \in R(S^1)[\![q]\!]$, where $R(S^1)$ in the complex representation ring for S^1.

The rigidity theorem (see [3] and references therein) asserts that the elliptic genus is rigid, i.e. each coefficient of $\operatorname{sign}_{S^1}(q, \mathcal{L}M)$ is constant as a character of S^1.

In a different cusp (the $\hat{A}$-cusp) $\Phi(M)$ expands as the following series of indices of twisted Dirac operators:

$$\begin{aligned} \Phi_0(M) &:= q^{-\frac{\dim M}{8}} \cdot \hat{A}\left(M, \bigotimes_{n=2m+1>0} \Lambda_{-q^n} TM \otimes \bigotimes_{n=2m>0} S_{q^n} TM\right) \\ &= q^{-\frac{\dim M}{8}} \cdot (\hat{A}(M) - \hat{A}(M, TM) \cdot q + \hat{A}(M, \Lambda^2 TM + TM) \cdot q^2 + \ldots). \end{aligned}$$

As in [9] we will study the equivariant expansion $\operatorname{sign}_{S^1}(q, \mathcal{L}M)$ evaluated at the involution $\sigma \in S^1$. Theorem 1.1 will follow from this and a "change of cusps" argument.

3. Proof of Theorem 1.1

Let M be a smooth closed connected *Spin*-manifold with smooth non-trivial S^1-action and let $\sigma \in S^1$ denote the element of order two. As in [9] we will study the equivariant elliptic genus $\mathrm{sign}_{S^1}(q, \mathcal{L}M)$ at σ via the Lefschetz fixed point formula [2].

Let $\mathrm{sign}_{S^1}(M, E)$ be an equivariant twisted signature occurring as a coefficient in $\mathrm{sign}_{S^1}(q, \mathcal{L}M)$. Using the Lefschetz fixed point formula the complex number $\mathrm{sign}_{S^1}(M, E)(\sigma)$ obtained by evaluating $\mathrm{sign}_{S^1}(M, E)$ at σ can be expressed as a sum of local data $a_{F,E}$ at the connected components F of the fixed point manifold $M^\sigma := \{p \in M \mid \sigma(p) = p\}$

$$\mathrm{sign}_{S^1}(M, E)(\sigma) = \sum_F a_{F,E}.$$

It is well known that the fixed point manifold M^σ is orientable (see for example [3, Lemma 10.1], and references therein). We fix an orientation for each connected component $F \subset M^\sigma$.

To describe the local datum $a_{F,E}$ consider the cohomology class $A_{F,E} \in H^*(F; \mathbb{Q})$ defined by the following expression:

$$\prod_i \left(x_i \cdot \frac{1 + e^{-x_i}}{1 - e^{-x_i}} \right) \cdot \prod_j \left(y_j \cdot \frac{1 + e^{-y_j}}{1 - e^{-y_j}} \right)^{-1} \cdot ch(E_{|F})(\sigma) \cdot e(\nu_F). \tag{1}$$

Here $\pm x_i$ (resp. $\pm y_j$) denote the formal roots of F (resp. of the normal bundle ν_F of F) for compatible orientations of F and ν_F, $e(\nu_F)$ is the Euler class of ν_F and $ch(E_{|F})$ denotes the equivariant Chern character of E restricted to F.

Then the local datum $a_{F,E}$ is obtained by evaluating the cohomology class $A_{F,E}$ on the fundamental cycle $[F]$

$$a_{F,E} = \langle A_{F,E}, [F] \rangle.$$

Lemma 3.1. *Let M and E be as above. If $H^k(M; \mathbb{Q}) = 0$ then $a_{F,E}$ vanishes for any connected component $F \subset M^\sigma$ of codimension k.*

Proof. Recall that the Euler class of the normal bundle of $i : F \hookrightarrow M$ is equal to $i^*(i_!(1))$, where $i_! : H^*(F; \mathbb{Z}) \to H^{*+k}(M; \mathbb{Z})$ denotes the push forward (or Gysin homomorphism) in cohomology for the oriented normal bundle ν_F. Since $H^k(M; \mathbb{Q}) = 0$ we see that $e(\nu_F)$ is a torsion class. Hence, $A_{F,E} = 0$ since it contains the Euler class $e(\nu_F)$ as a factor (see Eq. (1)). □

We shall now apply this observation to prove the following generalization of Theorem 1.1.

Theorem 3.2. *Let M be a Spin-manifold with $H^{4*}(M; \mathbb{Q}) = 0$ for $0 < * \leq r$. If M admits a smooth non-trivial S^1-action then the first $(r + 1)$ coefficients of $\Phi(M)$ vanish.*

Proof. Dividing out the kernel of the action we may assume that S^1 acts effectively. We may also assume that the dimension of M is divisible by 4. Let $\sigma \in S^1$ denote the element of order two. Recall that the S^1-action is called even if it lifts to the *Spin*-structure and odd otherwise. In the even case the codimension of all connected components of M^σ is divisible by 4 whereas in the odd case the codimensions are always $\equiv 2 \bmod 4$ (cf. [1, Lemma 2.4]). It is well known (see for example [9, p. 317]) that the elliptic genus vanishes for odd actions. So it suffices to restrict to the case where the dimension (and codimension) of each connected component $F \subset M^\sigma$ is divisible by 4.

Consider the expansion $\mathrm{sign}_{S^1}(q, \mathcal{L}M)$ of the S^1-equivariant elliptic genus in the signature cusp. The rigidity theorem [3] tells us that $\mathrm{sign}_{S^1}(q, \mathcal{L}M)(\sigma)$ is equal to the non-equivariant expansion $\mathrm{sign}(q, \mathcal{L}M)$. By the Lefschetz fixed point formula $\mathrm{sign}_{S^1}(q, \mathcal{L}M)(\sigma)$ is a sum of local contributions a_F at the connected components F of M^σ:

$$\mathrm{sign}(q, \mathcal{L}M) = \mathrm{sign}_{S^1}(q, \mathcal{L}M)(\sigma) = \sum_F a_F,$$

where $a_F = \sum_{n \geq 0} a_{F,E_n} \cdot q^n$.

Recall that each coefficient of the q-power series a_F is the local contribution in the Lefschetz fixed point formula of an equivariant twisted signature evaluated at $\sigma \in S^1$. Since $H^{4*}(M; \mathbb{Q}) = 0$ for $0 < * \leq r$ the contribution a_F vanishes if codim $F \leq 4r$ by Lemma 3.1. Hence,

$$\mathrm{sign}(q, \mathcal{L}M) = \sum_{\mathrm{codim}\ F > 4r} a_F.$$

As explained in [9] a_F can be identified with $\mathrm{sign}(q, \mathcal{L}(F \circ F))$, where $F \circ F$ denotes the transversal self-intersection (which is canonically oriented and unique up to oriented bordism). Hence,

$$\mathrm{sign}(q, \mathcal{L}M) = \sum_{\mathrm{codim}\ F \circ F > 8r} \mathrm{sign}(q, \mathcal{L}(F \circ F)).$$

Changing cusps one obtains

$$\Phi_0(M) = \sum_{\mathrm{codim}\ F \circ F > 8r} \Phi_0(F \circ F). \tag{2}$$

Note that each summand

$$\Phi_0(F \circ F) = q^{-\frac{\dim(F \circ F)}{8}} \cdot (\hat{A}(F \circ F) - \hat{A}(F \circ F, T(F \circ F)) \cdot q + \ldots)$$

of the right hand side of (2) has a pole of order $\leq \frac{\dim(F \circ F)}{8} < \frac{\dim M}{8} - r$. Comparing with the expansion on the left hand side

$$\Phi_0(M) = q^{-\frac{\dim M}{8}} \cdot (\hat{A}(M) - \hat{A}(M, TM) \cdot q + \ldots)$$

it follows that the first $(r + 1)$ coefficients of $\Phi_0(M)$ vanish. $\square$

Remark 3.3. Herrera and Herrera [7] have shown that the rigidity theorem also holds for manifolds with finite second homotopy group. Combining their result with the above argument shows that Theorem 3.2 remains true if one replaces the *Spin*-condition by the condition that the second homotopy group is finite.

4. Comparison with the Witten genus

In this section we show by example that for 8-connected manifolds Theorem 1.1 does not follow from the vanishing theorem for the Witten genus [12,6].

Let M be an $4k$-dimensional *Spin*-manifold with $\frac{p_1}{2}(M) = 0$. In [13] Witten introduced a genus, the so-called Witten genus, which is best thought of as the index of a hypothetical Dirac operator on the free loop space of M. The Witten genus is a bordism invariant which assigns to M

a modular form $\varphi_W(M)$ of weight $2k$ for $SL_2(\mathbb{Z})$ which we shall identify with its q-expansion. The latter can be described by the following series of indices of twisted Dirac operators:

$$\begin{aligned}\varphi_W(M) &= \hat{A}\left(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM\right) \cdot C_k \\ &= (\hat{A}(M) + \hat{A}(M, TM)\cdot q + \hat{A}(M, S^2TM + TM)\cdot q^2 + \ldots)\cdot C_k.\end{aligned}$$

Here $C_k := q^{-\frac{4k}{24}} \cdot \eta^{4k}$ and $\eta = q^{1/24} \cdot \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function.

It follows from [12] that the Witten genus vanishes on 4-connected manifolds with non-trivial smooth S^1-action (see [6] for related results). Applying this to the first two coefficients of the Witten genus we see that for any 4-connected manifold M with smooth non-trivial S^1-action $\hat{A}(M)$ and $\hat{A}(M, TM)$ must vanish.

Hence, the statement of Theorem 1.1 for 4-connected manifolds also follows from the vanishing theorem for the Witten genus. For 8-connected manifolds the situation changes as shown by the following.

Example 4.1. There is an 8-connected 28-dimensional manifold M with $\varphi_W(M) = 0$ but $\hat{A}(M, \Lambda^2 TM + TM) \neq 0$.

Remarks 4.2. 1. By Theorem 1.1 M does not admit a smooth non-trivial S^1-action.

2. The dimension is the smallest possible since any 8-connected manifold of dimension <28 with vanishing Witten genus is rationally zero bordant.

Outline of the construction of the example: The example will be obtained by applying surgery to a suitable combination of almost parallelizable manifolds. Let N_{4k} be a smooth closed almost parallelizable manifold with $\hat{A}(N_{4k}) = -\alpha_k \cdot num(\frac{B_{2k}}{4k})$, where B_{2k} is the $(2k)$th Bernoulli number, $num(\quad)$ denotes the numerator and α_k is one or two according as k is even or odd. Such manifolds which can be constructed via plumbing were considered by Kervaire and Milnor [10] (see [8, Section 6.4–6.5], for more details).

Let M' be the connected sum of N_{28} and $-(N_{16} \times N_{12})$, where the sign denotes opposite orientation. Since N_{4k} is almost parallelizable the stable normal bundle of M' is trivial over the 8-skeleton. Hence, we can use surgery to change M' inside its bordism class to an 8-connected manifold denoted by M.

To compute the Witten genus of M we first recall that for an almost parallelizable $4k$-dimensional manifold the Witten genus is equal to the $\hat{A}$-genus times the Eisenstein series E_{2k} (see [8, p. 89]). The first few terms of $\varphi_W(N_{4k})$ are given in the following table.

$2k$	$num(\frac{B_{2k}}{4k})$	E_{2k}	$\varphi_W(N_{4k})$
6	$+1$	$1-504q-\ldots$	$-2+1008q-\ldots$
8	-1	$1+480q+\ldots$	$+1+480q-\ldots$
14	$+1$	$1-24q-\ldots$	$-2+48q-\ldots$

Next we recall that the ring of modular forms of $SL_2(\mathbb{Z})$ is a polynomial ring generated by E_4 and E_6. This implies that a modular form of weight 14 vanishes if and only if its expansion has vanishing constant term. From the data above one computes that $\varphi_W(M) = 0$.

We now turn to the computation of the elliptic genus of M. The characteristic power series $Q(x) = x/f(x)$ of the elliptic genus Φ satisfies the differential equation $f'^2 = 1 - 2\frac{\delta}{\sqrt{\epsilon}} \cdot f^2 + f^4$, where δ and ϵ are modular forms for $\Gamma_0(2)$ of weight 2 and 4, respectively (see [8, Appendix I], for details). Note that the differential equation together with the normalization $f(x) = x + O(x^3)$ determines the odd power series $f(x)$ as well as $Q(x)$ and Φ.

For an almost parallelizable $4k$-dimensional manifold X the elliptic genus $\Phi(X)$ is equal to a constant s_k times the Pontrjagin number $\langle p_k(X), [X]\rangle$, where the s_k can be calculated from the characteristic power series using a formula of Cauchy:

$$1 - z\frac{\mathrm{d}}{\mathrm{d}z}\log Q(z) = \sum_{j=0}^{\infty}(-1)^j s_j \cdot z^j.$$

The Pontrjagin number $\langle p_k(X), [X]\rangle$ can be computed from $\hat{A}(X)$. Applying this information to the almost parallelizable manifolds N_{4k} one can compute $\Phi(M)$ as a polynomial in $\frac{\delta}{\sqrt{\epsilon}}$. To compute the expansion $\Phi_0(N_{4k})$ of $\Phi(N_{4k})$ in the $\hat{A}$-cusp one only has to replace $\frac{\delta}{\sqrt{\epsilon}}$ by its expansion in this cusp (again we refer the reader to [8, Appendix I]) for details). Doing the computation one obtains the following expansion for $\Phi_0(M) = \Phi_0(N_{28}) - \Phi_0(N_{16}) \cdot \Phi_0(N_{12})$:

$$\begin{aligned}\Phi_0(M) &:= q^{-\frac{28}{8}} \cdot (\hat{A}(M) - \hat{A}(M, TM) \cdot q + \hat{A}(M, \Lambda^2 TM + TM) \cdot q^2 + \ldots)\\ &= q^{-\frac{28}{8}} \cdot (-967\,680 \cdot q^2 - 127\,733\,760 \cdot q^3 + \ldots)\end{aligned}$$

Hence, $\hat{A}(M, \Lambda^2 TM + TM) \neq 0$.

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